Math 3280 Tutorial 7

Recall. 1. Find the distribution of $g(X)$. X is a cts r.v. with pdf f_X, $g: \mathbb{R} \to \mathbb{R}$.

$$F_{g(X)}(y) = P(g(X) \le y)$$

$$\Downarrow$$

$$f_{g(X)}(y) = \frac{d F_{g(X)}(y)}{dy}.$$

2. joint cdf of X, Y, $F(a,b) := P(X \le a, Y \le b)$, $a, b \in \mathbb{R}$.

$$F_X(a) = \lim_{b \to +\infty} F(a,b)$$

$$F_Y(b) = \lim_{a \to +\infty} F(a,b)$$

$$P(X > a, Y > b) = 1 - F(a,\infty) - F(\infty, b) + F(a,b)$$

3. For discrete r.v.s X, Y, the probability mass function

$$p(x,y) = P(X=x, Y=y)$$

$$p_X(x) = \sum_{y \in Y} p(x,y).$$

4. Two r.v.s are jointly continuous if exists

$$f: \mathbb{R}^2 \to [0,\infty).$$

s.t.

$$P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$$

$C \subseteq \mathbb{R}^2$.

$$P(X \le x, Y \le y) = \int^y \left(\int^x f(x,y)\,dx \right) dy$$

Example 1: If X is uniformly distributed on $(0,1)$. what is the probability density function of $Y=e^X$.

Solution:

$$f_X(x)=\begin{cases}1 & x\in(0,1)\\ 0 & \text{o.w.}\end{cases}$$

$Y=e^X\in(1,e)$.

If $x\le 1$, $F_Y(x)=P(Y\le x)=0$.

If $x\ge e$, $F_Y(x)=P(Y\le x)=1$.

If $\underline{1<x<e}$, $F_Y(x)=P(Y\le x)=P(e^X\le x)$

$\underline{0<\ln x<1}$

$$=P(X\le \ln x)$$
$$=\int_0^{\ln x} 1\,dt=\ln x$$

$$f_Y(x)=\begin{cases}0 & x\le 1\\ \frac{1}{x} & x\in(1,e)\\ 0 & x\ge e.\end{cases}$$

Example 2: The median of a r.v. with distribution F is defined to be the value of m such that $F(m)=\frac{1}{2}$. Find the median of X if X is

(a) uniformly distributed on (a,b)

(b) normal r.v. with μ,σ^2.

(c). exponential r.v. with λ.

Solution: (a) $X\sim U(a,b)$

$$f_X(x)=\begin{cases}\frac{1}{b-a} & x\in(a,b)\\ 0 & \text{o.w.}\end{cases}$$

$F_X(m) = \frac{1}{2}, \quad m \in (a, b)$

$F_X(m) = \int_a^m f_X(x)\,dx = \frac{m-a}{b-a} = \frac{1}{2} \Rightarrow m = \frac{a+b}{2}$.

(a). $X \sim N(\mu, \sigma^2)$, $\sigma > 0$.

$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

$\frac{1}{2} = F(m) = \int_{-\infty}^m f_X(x)\,dx = P(X \le m)$

$(Y = \frac{X-\mu}{\sigma}) = P(Y \le \frac{m-\mu}{\sigma})$

$Y \sim N(0,1) = \Phi(\frac{m-\mu}{\sigma})$.

cdf of Y.

$\Phi(0) = \frac{1}{2}$.

$\frac{m-\mu}{\sigma} = 0 \Rightarrow m = \mu$.

(c). $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0. \\ 0 & \text{o.w.} \end{cases}$

$\frac{1}{2} = F(m) = \int_0^m \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda m}$

$\Rightarrow \lambda m = \ln 2$

$m = \frac{\ln 2}{\lambda}$.

3. For any real numbers y, define y^+ by

$$y^+ = \begin{cases} y & y \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

Let c be a constant.

(a). Show that

cdf of $N(0,1)$

$$E[(Z-c)^+] = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} - c(1-\Phi(c)).$$

Z is a standard normal r.v.

(b) $Z \sim N(\mu, \sigma^2)$, find $E[(Z-c)^+]$.

Solution: (a) $Z \sim N(0,1)$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

$$E[(Z-c)^+] = \int_c^\infty (z-c) \cdot f_Z(z)\, dz, \quad z < c.\ (z-c)^+ = 0.$$

$$= \underbrace{\int_c^\infty z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}_{\int_c^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} d\frac{z^2}{2}} - \underbrace{c \int_c^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}_{= c\cdot(1-\Phi(c))}$$

$$= \frac{1}{\sqrt{2\pi}}\left(-e^{-\frac{z^2}{2}}\Big|_c^\infty\right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{c^2}{2}}$$

$$\underline{= \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} - c\cdot(1-\Phi(c))}.$$

(b) $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\cdot\sigma^2}}. \quad (\sigma \geq 0).$

$$E[(Z-c)^+] = \int_c^\infty (z-c) \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz$$

$$\overset{y=\frac{z-\mu}{\sigma}}{\underset{Y\sim N(0,1)}{=}} \int_{\frac{c-\mu}{\sigma}}^{\infty} (\sigma y+\mu-c)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy$$

$$= \sigma\int_{\frac{c-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}}\left(y-\frac{c-\mu}{\sigma}\right)\cdot e^{-\frac{y^2}{2}}\,dy$$

$$\overset{\text{denote } \frac{c-\mu}{\sigma}=a}{=} \sigma\cdot \underbrace{\int_{a}^{\infty} \frac{1}{\sqrt{2\pi}}(y-a)e^{-\frac{y^2}{2}}\,dy}$$

$$= \sigma\cdot E[(Y-a)^+]. \quad Y\sim N(0,1)$$

$$= \sigma\cdot\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{a^2}{2}} - a\cdot(1-\Phi(a))\right).$$

Example 4. The joint density function of X and Y is given by

$$f(x,y)=\begin{cases} 2e^{-x}\cdot e^{-2y} & , 0<x<\infty,\ 0<y<\infty \\ 0 & , \text{o.w} \end{cases}$$

(a) Find $P(X>1, Y<1)$

(b) $P(X<Y)$

(c) ...

(c) $P(X<a)$.

Solution: (a) $P(X>1, Y<1) = \int_0^1 \left(\int_1^\infty f(x,y)\,dx\right) dy$

$$= \int_0^1 \left(\int_1^\infty 2e^{-x} e^{-2y}\,dx\right) dy$$

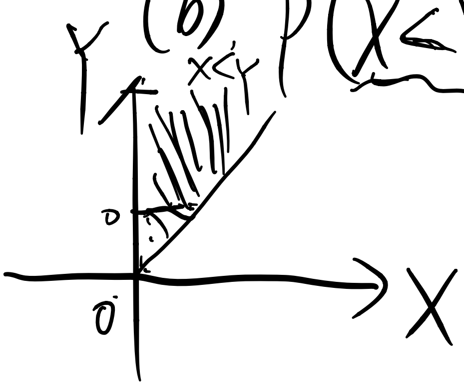
$\int_1^\infty e^{-x}\,dx = e^{-1}$

$$= \int_0^1 2e^{-2y} e^{-1}\,dy$$

$$= e^{-1} \int_0^1 2e^{-2y}\,dy$$

$$= e^{-1} \left(-e^{-2y} \Big|_0^1\right)$$

$$= e^{-1}(1-e^{-2}).$$



(b) $P(X<Y) = \int_0^\infty \left(\int_0^y f(x,y)\,dx\right) dy$

$$= \int_0^\infty \left(\underbrace{\int_0^y 2e^{-x} e^{-2y}\,dx}\right) dy$$

$2e^{-2y} \int_0^y e^{-x}\,dx = 2e^{-2y}(1-e^{-y})$

$$= \int_0^\infty 2e^{-2y}(1-e^{-y})\,dy$$

$$= 1 - \frac{2}{3} = \frac{1}{3}.$$

(c). $P(X<a) = \int_0^\infty \left(\int_0^a f(x,y)\,dx\right)dy$

$$= \int_0^\infty \left(\int_0^a 2e^{-x} e^{-2y}\,dx\right)dy$$

$$= \int_0^\infty 2\cdot e^{-2y} \cdot \underbrace{\left(\int_0^a e^{-x}\,dx\right)}_{1-e^{-a}}dy$$

$$= \int_0^\infty 2\cdot e^{-2y} \underline{(1-e^{-a})}\,dy$$

$$= (1-e^{-a}) \cdot \underbrace{\int_0^\infty 2e^{-2y}\,dy}_{\lambda=2,\ \text{exponential}} = 1$$

$$= 1-e^{-a}.$$